Student Number:

Set:

Shore

Year 12
Mathematics - Extension 2
Trial Examination
2009

General Instructions

* Reading time – 5 minutes
* Working time – 3 hours
* Write using black or blue pen
* Board-approved calculators may be used
* All necessary working should be shown in every question
* A table of standard integrals is attached on the final page

Note: Any time you have remaining should be spent revising your answers.

Total marks - 120

* Attempt Questions 1 - 8
* All questions are of equal value
* Start each question in a new writing booklet
* Write your examination number on the front cover of each booklet to be handed in
* If you do not attempt a question, submit a blank booklet with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total marks - 120
Attempt Questions 1 - 8

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Find the indefinite integrals:

(i) $\int \sec^4 x \, dx$ **2**

(ii) $\int \sqrt{1-x^2} \, dx$ **4**

(b) Consider the definite integral $I_n = \int_0^2 \frac{x^n}{x^3+1} dx$.

(i) Show that $I_2 = \frac{2}{3}\log_e 3$. **2**

(ii) Using your knowledge of factorisation and without evaluating more than one integral, show that

$$I_2 - I_1 + I_0 = \log_e 3$$ **2**

(iii) Using a similar approach to that used in (ii), show that

$$I_1 + I_0 = \frac{\pi}{\sqrt{3}}.$$ **3**

(iv) Using the above results or otherwise find the exact value of I_0. **2**

Question 2 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Make neat sketches of the following, showing all intercepts and asymptotes. There is no need to use calculus.

(i) $y = x^2(x-2)(x-3)$ **2**

(ii) $y = \dfrac{1}{x^2(x-2)(x-3)}$ **2**

(iii) $y = \dfrac{x^2}{(x-2)(x-3)}$ **2**

(iv) $y = x\sqrt{(x-2)(x-3)}$ **2**

(v) $y = x^2|x-2|(x-3)$ **2**

(b) Consider the equation $e^{2x} = k\sqrt{x}$.

(i) Explain why this equation has no solutions when $k \le 0$. **1**

(ii) Find the value of k for which the equation has exactly one real solution. **4**

Question 3 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Given $z = 1 - 2i$ is a complex root of the quadratic equation $z^2 + (1+i)z + k = 0$, find the other root and the value of k. **3**

(b) Find all complex numbers $z = a + bi$, where a and b are real such that $|z|^2 - iz = 16 - 2i$. **3**

(c) Consider all complex numbers z such that $\arg\left(\frac{z-1}{z-i}\right) = \frac{\pi}{4}$

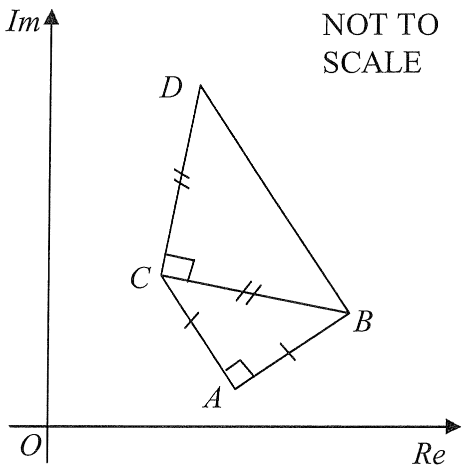
(i) Make a neat sketch of the locus of z showing important features. **2**

(ii) Determine the exact maximum value of $|z|$. **1**

(iii) Determine (in radians correct to 3 significant figures) the maximum value of $\arg(z+1)$. **3**

Question 3 continues on page 5

(d)



In the diagram, the points A, B, C and D represent the complex numbers z_1, z_2, z_3 and z_4 respectively. Both ΔABC and ΔBCD are right angled isosceles triangles as shown.

(i) Show that the complex number z_3 can be written as

$$z_3 = (1-i)z_1 + iz_2 .$$ **1**

(ii) Hence express the complex number z_4 in terms of z_1 and z_2, giving your answer in simplest form. **2**

End of Question 3

Question 4 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Use Mathematical Induction to prove De Moivre's Theorem, ie

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n. **4**

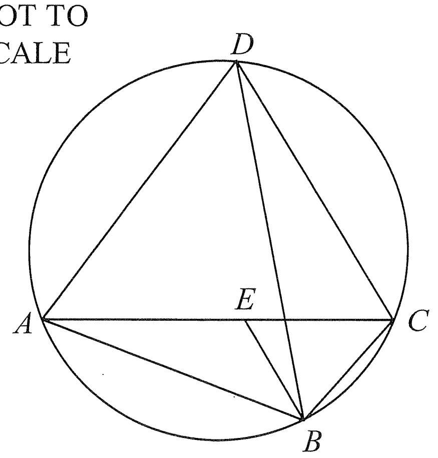
(b) The equation $x^3 + 3px - 1 = 0$, where p is real, has roots α, β and γ.

(i) Show that the monic cubic equation, with coefficients in terms of p, whose roots are α^2, β^2 and γ^2 is

$$y^3 + 6py^2 + 9p^2y - 1 = 0.$$ **2**

(ii) Hence or otherwise obtain the monic cubic equation, with coefficients in terms of p, whose roots are $\frac{\beta\gamma}{\alpha}$, $\frac{\gamma\alpha}{\beta}$ and $\frac{\alpha\beta}{\gamma}$. **3**

(c) NOT TO SCALE



The figure shows a cyclic quadrilateral $ABCD$ with diagonals AC and BD.

E is a point on AC such that $\angle ABE = \angle DBC$.

Make a neat copy of the diagram in your answer booklet.

(i) Prove that $\Delta ABE \mathbin{|||} \Delta DBC$. **2**

(ii) Prove that $\Delta ABD \mathbin{|||} \Delta EBC$. **2**

(iii) Hence prove Ptolemy's Theorem, which is:

$$AB.DC + AD.BC = AC.BD$$ **2**

Question 5 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Determine the eccentricity of the ellipse with equation $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$ and then make a neat sketch of the curve, clearly showing the coordinates of the foci and the equations of the directrices. **4**

(b) A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N.

(i) Make a neat sketch illustrating the information above. **1**

(ii) Show that the equation of the normal at P is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$ **2**

(iii) Show that QN is perpendicular to the asymptote. **2**

(c) $P\left(p,\dfrac{1}{p}\right)$ and $Q\left(q,\dfrac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$ such that the chord PQ passes through the point $A(0,2)$. M is the midpoint of PQ.

(i) Show that PQ has equation $x+pqy-(p+q)=0$ and hence deduce that $p+q=2pq$. **3**

(ii) You may assume that the tangent to $xy=1$ at the point $(1,1)$ passes through A. Determine the locus of M, being sure to state any restrictions on the domain. **3**

Question 6 **(15 marks)** Use a SEPARATE writing booklet **Marks**

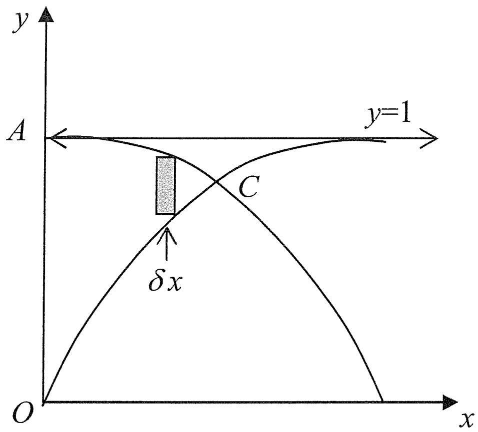
(a) The base of a solid is the region enclosed by the circle $x^2 + y^2 = 4$.
Any cross sections of the solid formed by a plane perpendicular to the x-axis are equilateral triangles. Find the exact volume of the solid. **4**

(b) (i) Make a neat sketch of the region enclosed between the curve $y = (x-3)^2$ and the line $3x + y - 9 = 0$. Be sure to mark in the points of intersection. **2**

(ii) The shaded region in (i) is rotated about the line $x = 3$. Use the method of cylindrical shells to find the exact volume of the solid generated. **3**

Question 6 continues on page 9

(c) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y-axis at A, and C is the first point of intersection of the two graphs to the right of the y-axis. The region OAC is to be rotated about the line $y = 1$.



NOT TO SCALE

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by

$$\delta V = \pi\left(2\cos x - 2\sin x - \cos 2x\right)\delta x.$$ **2**

(iii) Hence find the exact volume of the solid formed when the region OAC is rotated about the line $y = 1$. **3**

End of Question 6

Question 7 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) (i) Prove that $\tan^{-1}(n+1)-\tan^{-1}(n)=\cot^{-1}\left(1+n+n^2\right)$ **2**

(ii) Hence find the sum of the finite series

$$\cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+\ldots\ldots+\cot^{-1}\left(1+n+n^2\right)$$

Give your answer in simplest form. **2**

(b) You are given that for the complex number $z=\cos\theta+i\sin\theta$ and for positive integers n, the following results are true:

$$z^n+\frac{1}{z^n}=2\cos n\theta \quad \text{and} \quad z^n-\frac{1}{z^n}=2i\sin n\theta$$

(i) Expand $\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4$ and hence show that

$$4\cos^4\theta+4\sin^4\theta=\cos 4\theta+3$$ **3**

(ii) By letting $x=\cos\theta$, show that the equation

$8x^4+8\left(1-x^2\right)^2=7$ has roots $x=\pm\cos\frac{\pi}{12},\ \pm\cos\frac{5\pi}{12}$. **3**

(iii) Deduce that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=\frac{1}{4}$ and $\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}=\sqrt{\frac{3}{2}}$. **3**

(iv) Hence or otherwise express $\cos\frac{\pi}{12}$ in surd form. **2**

Question 8 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Six letters are chosen from the word AUSTRALIA. These six letters are then placed alongside one another to form a six letter arrangement. Find the number of distinct six letter arrangements which are possible, considering all choices. **4**

(b) It is given that for three positive real numbers a, b and c,

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

If we also know that $a+b+c=1$, prove that

(i) $\dfrac{1}{abc} \geq 27$ **1**

(ii) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

(i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$ for $n \geq 2$. **3**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \dfrac{\pi (2n)!}{2^{2n+1} (n!)^2}$ **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

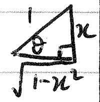
Note: $\ln x = \log_e x, x > 0$

1 (a) (i) $I = \int \sec^4 x \, dx$

$= \int \sec^2 x(1+\tan^2 x)\, dx$

$= \tan x + \frac{1}{3}\tan^3 x + C$ 2

(ii) $I = \int \sqrt{1-x^2}\, dx$ let $x = \sin\theta$, $dx = \cos\theta \, d\theta$



$= \int \sqrt{1-\sin^2\theta}\cos\theta \, d\theta$

$= \int \cos^2\theta \, d\theta$

$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$

$= \frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2} + C$ 4

(b) $I_n = \int_0^2 \frac{x^n}{x^3+1}\, dx$

(i) $I_2 = \int_0^2 \frac{x^2}{x^3+1}\, dx$

$= \left[\frac{1}{3}\log_e(x^3+1)\right]_0^2$

$= \frac{1}{3}(\log_e 9 - \log_e 1)$

$= \frac{2}{3}\log_e 3$ 2

(ii) $I_2 - I_1 + I_0 = \int_0^2 \frac{x^2-x+1}{x^3+1}\, dx$

$= \int_0^2 \frac{1}{x+1}\, dx$

$= \left[\log_e(x+1)\right]_0^2$

$= \log_e 3$ 2

(iii) $I_1 + I_0 = \int_0^2 \frac{x+1}{x^3+1}\, dx$

$= \int_0^2 \frac{1}{x^2-x+1}\, dx$

$= \int_0^2 \frac{1}{\frac{3}{4}+(x-\frac{1}{2})^2}\, dx$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^2$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right)$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$

$= \frac{\pi}{\sqrt{3}}$ 3

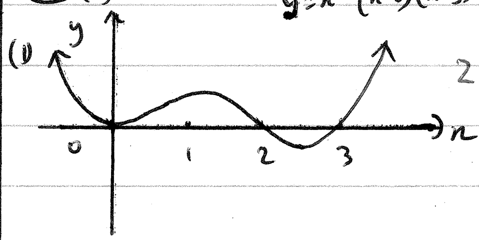
(iv) $I_0 = \frac{1}{2}\left[I_1+I_0+I_2-I_1+I_0-I_2\right]$

$= \frac{1}{2}\left[\frac{\pi}{\sqrt{3}} + \log_e 3 - \frac{2}{3}\log_e 3\right]$

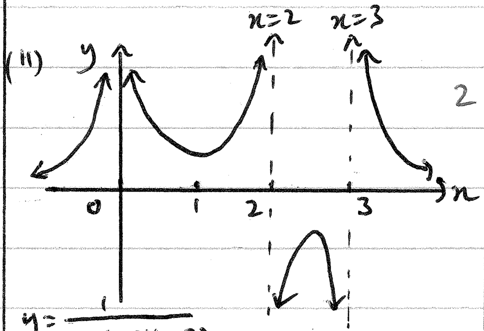
$= \frac{\pi}{2\sqrt{3}} + \frac{1}{6}\log_e 3$ 2

2 (a) $y = x^2(x-2)(x-3)$

(i) 2

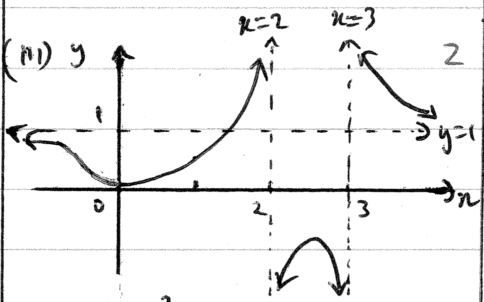


(ii) 2



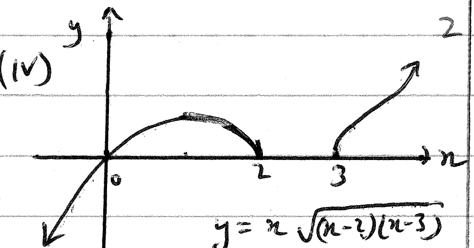
$y = \frac{1}{x^2(x-2)(x-3)}$

(iii) 2



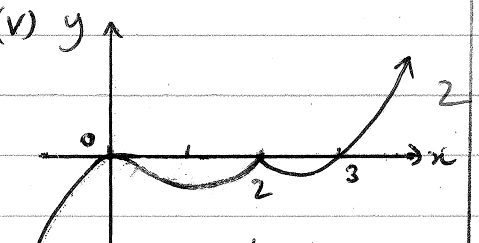
$y = \frac{x^2}{(x-2)(x-3)}$

(iv) 2



$y = x\sqrt{(x-2)(x-3)}$

(v) 2



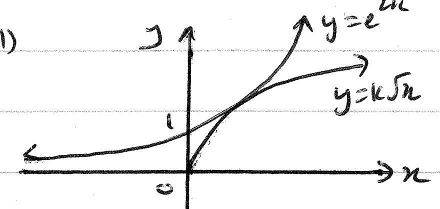
$y = x^2|x-2|(x-3)$

(b) $e^{2x} = k\sqrt{x}$

(i) $e^{2x} > 0$ for all x, $\sqrt{x} \geq 0$

so if $k \leq 0$, $e^{2x} \neq k\sqrt{x}$. 1

(ii)



For one real solution $y = e^{2x}$ and $y = k\sqrt{x}$ have one point of int. and share a common tangent.

$y = e^{2x}$, $y = k\sqrt{x}$

$y' = 2e^{2x}$, $y' = \frac{k}{2\sqrt{x}}$

$\Rightarrow 2k\sqrt{x} = \frac{k}{2\sqrt{x}}$

$\Rightarrow 4x = 1 \Rightarrow x = \frac{1}{4}$

$\Rightarrow e^{\frac{1}{2}} = k\sqrt{\frac{1}{4}}$

$\Rightarrow k = 2\sqrt{e}$ 4

3 (a) $z^2 + (1+i)z + k = 0$

One root $1-2i$, let other be α

$\alpha + 1 - 2i = -1 - i \Rightarrow \alpha = -2+i$ 1

$\therefore k = (1-2i)(-2+i)$

$= -2 + i + 4i + 2$

$= 5i$ 2

(b) $|z|^2 - iz = 16 - 2i$, $z = a+bi$

$a^2+b^2 - ai + b = 16 - 2i$

$\Rightarrow a^2+b^2+b = 16$ and $a = 2$

$\therefore 2^2 + b^2 + b = 16$

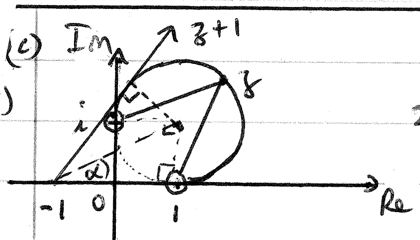
$b^2 + b - 12 = 0$

$(b+4)(b-3) = 0$

$b = -4$ or 3 3

$\therefore z = 2+3i$ or $2-4i$

(c) (i)



2

(ii) $\text{Max } |z| = \sqrt{2}+1$ 1

(iii) $\text{Max } \arg(z+1) = 2\tan^{-1}\frac{1}{2}$

$= 0.868$ 3

(d) (i) $\vec{AC} = i \cdot \vec{AB}$

$z_3 - z_1 = i(z_2 - z_1)$

$z_3 = (1-i)z_1 + iz_2$ 1

(ii) $\vec{CD} = i \cdot \vec{CB}$

$z_4 - z_3 = i(z_2 - z_3)$

$z_4 = iz_2 + (1-i)z_3$

$= iz_2 + (1-i)^2 z_1 + i(1-i)z_2$

$= -2iz_1 + (1+2i)z_2$ 2

4 (a) RTP $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$n=1$, LHS $= \cos\theta + i\sin\theta =$ RHS $\therefore$ True

Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Now $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$

$= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$\therefore$ If true for $n=k$, also true for $n=k+1$

Since true when $n=1$, also when $n=1+1=2$ and so on

4

(b) (i) $x^3 + 3px - 1 = 0$ has roots α, β, γ

Let $y = x^2$, $x = \pm\sqrt{y}$

$(\pm\sqrt{y})^3 + 3p(\pm\sqrt{y}) - 1 = 0$

$\pm\sqrt{y}(y + 3p) = 1$

$y(y^2 + 6py + 9p^2) = 1$

$y^3 + 6py^2 + 9p^2y - 1 = 0$ 2

has roots $\alpha^2, \beta^2, \gamma^2$.

(ii) $\alpha\beta\gamma = 1 \Rightarrow \frac{\beta\gamma}{\alpha} = \frac{1}{\alpha^2}$ etc

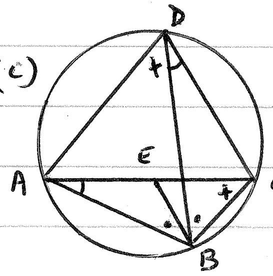
$\therefore$ Let $z = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

$(\frac{1}{z})^3 + 6p(\frac{1}{z})^2 + 9p^2(\frac{1}{z}) - 1 = 0$

$1 + 6pz + 9p^2z^2 - z^3 = 0$ 3

$\therefore z^3 - 9p^2z^2 - 6pz - 1 = 0$ roots $\frac{\beta\gamma}{\alpha}$, etc.

(c)



Given $\angle ABE = \angle DBC$

(i) $\angle ABE = \angle DBC$ (Given) 2

$\angle BAE = \angle BDC$ ($\angle$'s in same segment)

$\therefore \Delta ABE ||| \Delta DBC$ (2 pairs of equal $\angle$'s)

(ii) $\angle ABD = \angle ABE + \angle EBD$

$\angle EBC = \angle DBC + \angle EBD$

$\therefore \angle ABD = \angle EBC$ 2

Also $\angle ADB = \angle ECB$ ($\angle$'s in same segment)

$\therefore \Delta ABD ||| \Delta EBC$ (2 pairs of equal $\angle$'s)

(iii) $\frac{AB}{DB} = \frac{AE}{DC}$ (matching sides in sim Δ's)

$\Rightarrow AB \cdot DC = AE \cdot DB$ (1)

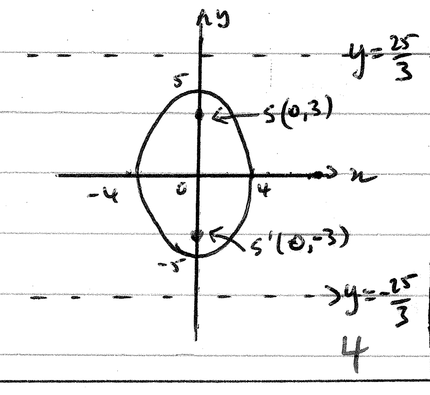
Also $\frac{AD}{EC} = \frac{BD}{BC}$ (matching sides in sim Δ's)

$\Rightarrow AD \cdot BC = EC \cdot BD$ (2)

(1)+(2) $AB \cdot DC + AD \cdot BC = (AE + EC)BD$

$= AC \cdot BD$ 2

5 (a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$, $e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$



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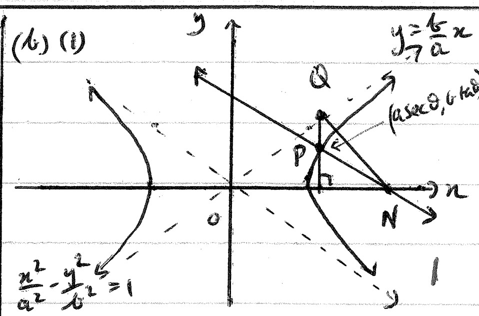
(b) (i)



1

(ii) At P, $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

$y = b\tan\theta$, $dy = b\sec^2\theta\, d\theta$

$\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$

Normal: $y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$by\sec\theta - b^2\sec\theta\tan\theta = -ax\tan\theta + a^2\sec\theta\tan\theta$

$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ 2

(iii) At Q, $x = a\sec\theta$, $y = b\sec\theta$

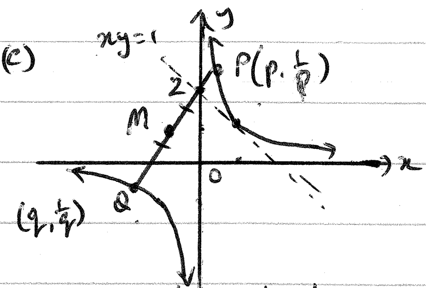
At N, $y = 0 \Rightarrow x = \frac{\sec\theta}{a}(a^2 + b^2)$

Grad QN $= \frac{0 - b\sec\theta}{a\sec\theta + \frac{b^2}{a}\sec\theta - a\sec\theta}$

$= -\frac{a}{b}$ 2

Asymptote has grad $\frac{b}{a} \Rightarrow \perp$ to QN

(c)



(i) PQ: $\frac{y - \frac{1}{p}}{x - p} = \frac{\frac{1}{q} - \frac{1}{p}}{q - p}$ $(\times pq)$

$(pqy - q)(q - p) = (x - p)(p - q)$

$\Rightarrow x + pqy - (p + q) = 0$ 2

subst $(0, 2) \Rightarrow 2pq = p + q$ 1

(ii) M is $\left(\frac{1}{2}(p+q), \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right)$

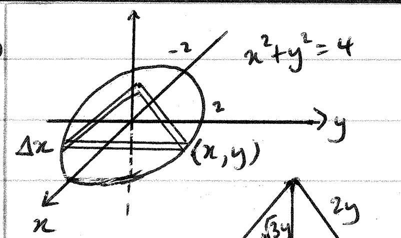
ie $\left(\frac{1}{2}(p+q), \frac{q+p}{2pq}\right)$

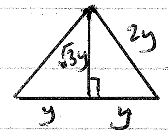
ie $\left(\frac{1}{2}(p+q), 1\right)$

$\therefore$ Locus of M is $y = 1$ 3

but only for $x < 0, x > 1$

6 (a)





$\Delta V = \frac{1}{2}\times 2y \times \sqrt{3}y \times \Delta x$

$= \sqrt{3}(4-x^2)\Delta x$

$V = \lim_{\Delta x\to 0} \sum_{-2}^{2} \sqrt{3}(4-x^2)\Delta x$

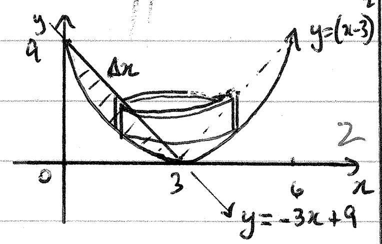
$= 2\sqrt{3}\int_0^2 (4-x^2)\,dx$

$= 2\sqrt{3}\left[4x - \frac{1}{3}x^3\right]_0^2$

$= 2\sqrt{3}\left(8 - 2\frac{2}{3}\right)$

$= \frac{32\sqrt{3}}{3}$ cubic units. 4

(b) (i)



2

(ii) $\Delta V = 2\pi(3-x)(-3x+9-(x-3)^2)\Delta x$

$= 2\pi(3-x)(3x-x^2)\Delta x$

$= 2\pi(9x-6x^2+x^3)\Delta x$

$V = \lim_{\Delta x\to 0}\sum_{0}^{3} 2\pi(9x-6x^2+x^3)\Delta x$

$= 2\pi\int_0^3 (9x-6x^2+x^3)\,dx$

$= 2\pi\left[\frac{9}{2}x^2 - 2x^3 + \frac{1}{4}x^4\right]_0^3$

$= \frac{27\pi}{2}$ cubic units. 3

(c) (i) C is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ 1

(ii) $\Delta V = \pi\left[(1-\sin x)^2 - (1-\cos x)^2\right]\Delta x$

$= \pi\left[1 - 2\sin x + \sin^2 x - (1 - 2\cos x + \cos^2 x)\right]\Delta x$

$= \pi\left[2\cos x - 2\sin x - (\cos^2 x - \sin^2 x)\right]\Delta x$

$= \pi\left[2\cos x - 2\sin x - \cos 2x\right]\Delta x$ 2

(iii) $V = \lim_{\Delta x\to 0}\sum_{0}^{\frac{\pi}{4}} \pi\left[2\cos x - 2\sin x - \cos 2x\right]\Delta x$

$= \pi\int_0^{\frac{\pi}{4}} (2\cos x - 2\sin x - \cos 2x)\,dx$

$= \pi\left[2\sin x + 2\cos x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \pi\left[\sqrt{2}+\sqrt{2}-\frac{1}{2} - (0+2-0)\right]$

$= \frac{\pi}{2}(4\sqrt{2}-5)$ cubic units. 3

7 (a)(i) Let $\alpha = \tan^{-1}(n+1)$, $\beta = \tan^{-1} n$

$\Rightarrow \tan\alpha = n+1$, $\tan\beta = n$

Now $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{n+1-n}{1+(n+1)n}$

$\therefore \cot(\alpha-\beta) = 1+n+n^2$

$\therefore \alpha - \beta = \cot^{-1}(1+n+n^2)$ 2

$\therefore \tan^{-1}(n+1) - \tan^{-1} n = \cot^{-1}(1+n+n^2)$

(ii) $\cot^{-1}(3) + \cot^{-1}(7) + \cdots + \cot^{-1}(1+n+n^2)$

$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \cdots + (\tan^{-1}(n+1) - \tan^{-1}n)$

$= \tan^{-1}(n+1) - \frac{\pi}{4}$ 2

(b) (i) $\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4$

$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} + z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$

$= 2\left(z^4 + \frac{1}{z^4}\right) + 12$

3

$\therefore (2\cos\theta)^4 + (2i\sin\theta)^4 = 4\cos 4\theta + 12$

$\therefore 16\cos^4\theta + 16\sin^4\theta = 4\cos 4\theta + 12$

$4\cos^4\theta + 4\sin^4\theta = \cos 4\theta + 3$

(ii) $8x^4 + 8(1-x^2)^2 = 7$

Let $x = \cos\theta$

$8\cos^4\theta + 8(1-\cos^2\theta)^2 = 7$

$8\cos^4\theta + 8\sin^4\theta = 7$

$\Rightarrow 2(\cos 4\theta + 3) = 7$

$\Rightarrow \cos 4\theta = \frac{1}{2}$

$\therefore 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$

$\therefore$ Solutions $x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

ie $x = \pm\cos\frac{\pi}{12}, \pm\cos\frac{5\pi}{12}$ 3

(iii) $8x^4 + 8(1-x^2)^2 = 7$

$8x^4 + 8 - 16x^2 + 8x^4 = 7$

$16x^4 - 16x^2 + 1 = 0$ $(x^2 = m)$

$16m^2 - 16m + 1 = 0$

has roots $\cos^2\frac{\pi}{12}$, $\cos^2\frac{5\pi}{12}$

$\therefore \cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12} = \frac{1}{16}$

$\therefore \cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$ $\cos\frac{\pi}{12} > 0$, $\cos\frac{5\pi}{12} > 0$

Also

$\left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{5\pi}{12}$

$= \frac{-(-16)}{16} + 2\times\frac{1}{4}$

$= \frac{3}{2}$ 3

$\therefore \cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \sqrt{\frac{3}{2}}$

(iv) $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$ are solutions to the equation:

$x^2 - \sqrt{\frac{3}{2}}x + \frac{1}{4} = 0$

ie $4x^2 - 2\sqrt{6}x + 1 = 0$

$x = \frac{2\sqrt{6} \pm \sqrt{24 - 4\times4\times1}}{2\times4}$

$= \frac{2\sqrt{6} \pm 2\sqrt{2}}{8}$

$= \frac{\sqrt{6} \pm \sqrt{2}}{4}$ 2

$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$, $\cos\frac{\pi}{12} > \cos\frac{5\pi}{12}$

8 (a) AUSTRALIA

No. of words = No with diff letters + No with 2 A's + No with 3 A's

$$= {}^7C_6 \times 6! + {}^6C_4 \times \frac{6!}{2!} + {}^6C_3 \times \frac{6!}{3!}$$

$$= 5040 + 5400 + 2400$$

$$= \underline{\underline{12840}}$$ 4

(b) $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$, $a+b+c=1$

$a>0, b>0, c>0$

(i) $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

$\Rightarrow \frac{1}{3} \geq \sqrt[3]{abc}$

$\Rightarrow \frac{1}{27} \geq abc$

$\Rightarrow 27 \leq \frac{1}{abc}$

$\Rightarrow \underline{\underline{\frac{1}{abc} \geq 27}}$ 1

(ii) $\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{a}\frac{1}{b}\frac{1}{c}}$

$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3 \times \sqrt[3]{\frac{1}{abc}}$

$\geq 3 \times \sqrt[3]{27}$

$\geq 3 \times 3$

$\underline{\underline{\geq 9}}$ 2

(iii) From (ii)

$\frac{bc+ac+ab}{abc} \geq 9$

$\therefore ab+bc+ac \geq 9abc$ (*)

Now $(1-a)(1-b)(1-c)$

$=(1-a-b+ab)(1-c)$

$=1-a-b+ab-c+ac+bc-abc$

$=1-(a+b+c)+(ab+bc+ac)-abc$

$=(ab+bc+ac)-abc$

$\geq 9abc - abc$ from (*)

$\geq \underline{\underline{8abc}}$ 2

(c) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}x . \sin x \, dx$

Let $u = \sin^{n-1}x$ $\quad v' = \sin x$

$u' = (n-1)\sin^{n-2}x\cos x$ $\quad v = -\cos x$

$I_n = \left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x \, dx$

$= 0 - 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x(1-\sin^2 x)\, dx$

$= (n-1)\left[I_{n-2} - I_n\right]$

$\therefore [1+(n-1)] I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$\therefore \underline{\underline{I_n = \left(\frac{n-1}{n}\right) I_{n-2}}}$ 3

(ii) $\int_0^{\frac{\pi}{2}} \sin^{2n} dx = I_{2n}$

$= \frac{2n-1}{2n} . I_{2n-2}$

$= \frac{(2n-1)(2n-3)}{2n(2n-2)} I_{2n-4}$

$= \frac{(2n-1)(2n-3)(2n-5)\dots 3\times 1}{2n(2n-2)(2n-4)\dots 4\times 2} I_0$

$= \frac{2n(2n-1)(2n-2)(2n-3)\dots\times 3\times 2\times 1}{[2n(2n-2)(2n-4)\dots\dots 4\times 2]^2} I_0$

$= \frac{(2n)!}{(2^n)^2[n(n-1)(n-2)\dots 2\times 1]^2} I_0$

$= \frac{(2n)!}{2^{2n}(n!)^2} \int_0^{\frac{\pi}{2}} 1 \cdot dx$

$= \frac{(2n)!}{2^{2n}(n!)^2} [x]_0^{\frac{\pi}{2}}$

$= \frac{(2n)!}{2^{2n}(n!)^2} \times \frac{\pi}{2}$

$= \underline{\underline{\frac{\pi (2n)!}{2^{2n+1}(n!)^2}}}$ 3